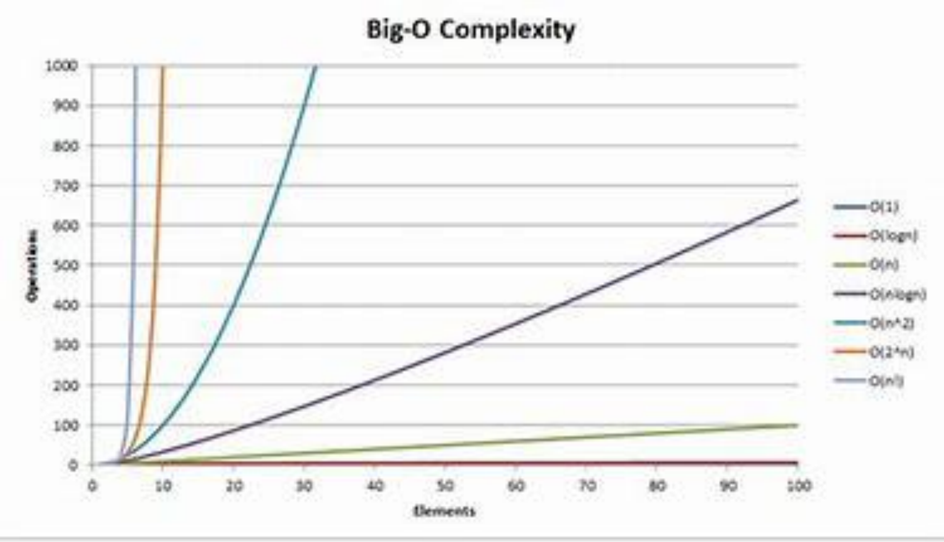



# Asymptotic Analysis

## Big–Oh: Definition

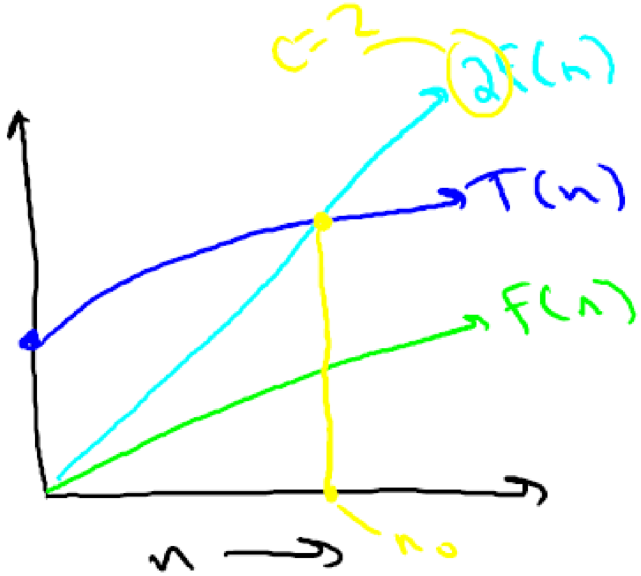
Design and Analysis of Algorithms I

# Big-Oh: English Definition

Let T(n) = function on n = 1,2,3,...
[usually, the worst-case running time of an algorithm]

Q : When is T(n) = O(f(n)) ?
A : if eventually (for all sufficiently large n), T(n) is bounded above by a constant multiple of f(n)

# Big-Oh: Formal Definition




$Picture \;\; T(n) = O(f(n))$

<u>Formal Definition</u> : T(n) = O(f(n)) if and only if there exist constants $c, n_0 > 0$ such that

$$T(n) \leq c \cdot f(n)$$

For all $n \geq n_0$

<u>Warning</u> : $c, n_0$ cannot depend on n

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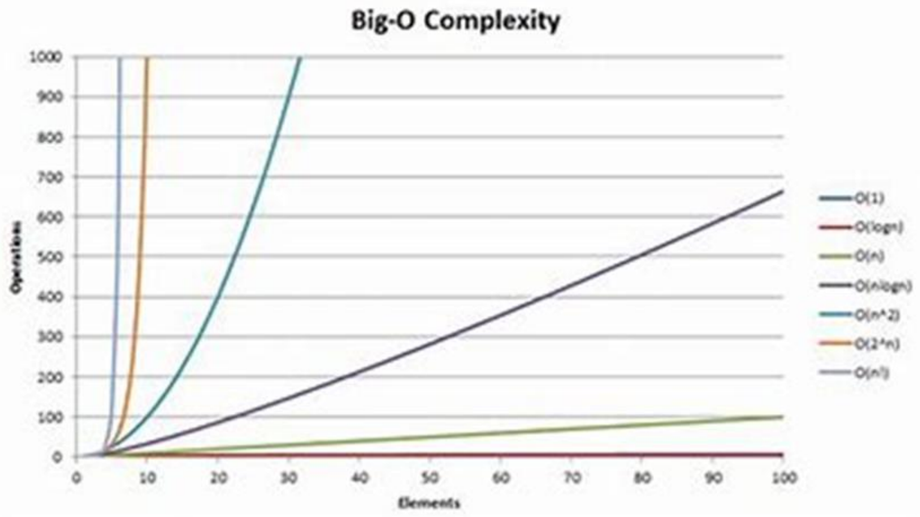



# Asymptotic Analysis

## Big-Oh: Basic Examples

Design and Analysis of Algorithms I

# Example #1

Claim : if $T(n) = a_k n^k + ... + a_1 n + a_0$ then

$$T(n) = O(n^k)$$

Proof : Choose $n_0 = 1$ and $c = |a_k| + |a_{k-1}| + .. + |a_1| + |a_0|$

Need to show that $\forall n \geq 1, T(n) \leq c \cdot n^k$

We have, for every $n \geq 1$,

$$\begin{aligned} T(n) &\leq |a_k|n^k + ... + |a_1|n + |a_0| \\ &\leq |a_k|n^k + ... + |a_1|n^k + |a_0|n^k \\ &= c \cdot n^k \end{aligned}$$

# Example #2

Claim : for every $k \geq 1,\ n^k$ is not $O(n^{k-1})$

Proof : by contradiction. Suppose $n^k = O(n^{k-1})$

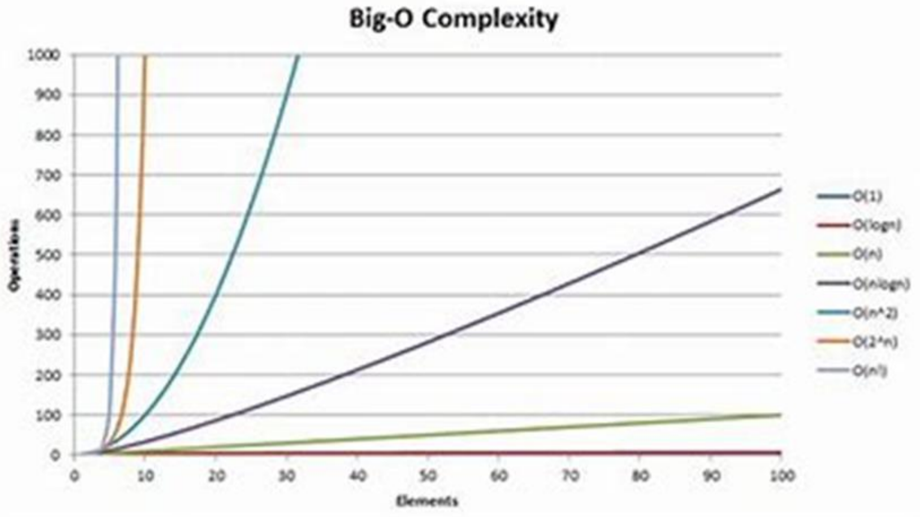
Then there exist constants $c, n_0$ such that

$$n^k \leq c \cdot n^{k-1} \quad \forall n \geq n_0$$

But then [cancelling $n^{k-1}$ from both sides]:

$$n \leq c \quad \forall n \geq n_0$$

Which is clearly False [contradict i o n ]




# Asymptotic Analysis

## Big Oh: Relatives (Omega & Theta)

Design and Analysis of Algorithms I

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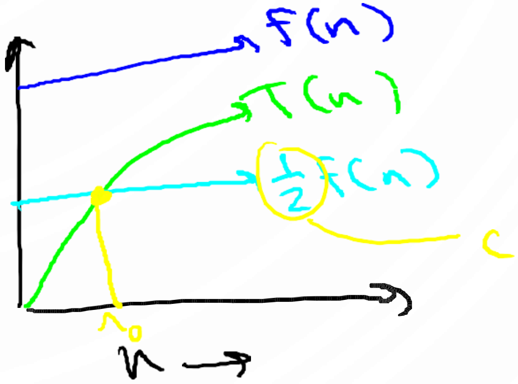
# OMEGA NOTATION

<u>Definition</u> : $T(n) = \Omega(f(n))$

If and only if there exist constants $c, n_0$ such that

$$T(n) \geq c \cdot f(n) \quad \forall n \geq n_0 \,.$$

Picture




$$T(n) = \Omega(f(n))$$

# Theta Notation

<u>Defini4on</u> : $T(n) = \theta(f(n))$ if and only if
$T(n) = O(f(n))$ and $T(n) = \Omega(f(n))$

<u>Equivalent</u> : there exist constants $c_1, c_2, n_0$ such that

$$c_1 f(n) \leq T(n) \leq c_2 f(n)$$

$$\forall n \geq n_0$$

NEXTCORE AI

Let $T(n) = \frac{1}{2}n^2 + 3n$ . Which of the following statements are true ? (Check all that apply.)

- ☐ $T(n) = O(n)$.
- ☐ $T(n) = \Omega(n)$. $[n_0 = 1, c = 1/2]$
- ☐ $T(n) = \Theta(n^2)$. $[n_0 = 1, c_1 = 1/2, c_2 = 4]$
- ☐ $T(n) = O(n^3)$. $[n_0 = 1, c = 4]$

# Little-Oh Notation

<u>Definition</u> : $T(n) = o(f(n))$ if and only if for all constants c>0, there exists a constant $n_0$ such that

$$T(n) \leq c \cdot f(n) \quad \forall n \geq n_0$$

<u>Exercise</u> : $\forall k \geq 1, n^{k-1} = o(n^k)$

# Where Does Notation Come From?

- "On the basis of the issues discussed here, I propose that members of SIGACT, and editors of compter science and mathematics journals, adopt the $O$, $\Omega$, and $\Theta$ notations as defined above, unless a better alternative can be found reasonably soon".

*-D. E. Knuth, "Big Omicron and Big Omega and Big Theta", SIGACT News, 1976. Reprinted in "Selected Papers on Analysis of Algorithms."*

# Asymptotic Analysis

## Additional Examples

Design and Analysis of Algorithms I

# EXAMPLE #1

Claim : $2^{n+10} = O(2^n)$

Proof : need to pick constants $c, n_0$ such that

$$(*) \quad 2^{n+10} \leq c \cdot 2^n \quad n \geq n_0$$

Note : $2^{n+10} = 2^{10} \times 2^n = (1024) \times 2^n$

So if we choose $c = 1024, n_0 = 1$ then (*) holds.

Q.E.D

# EXAMPLE #2

Claim : $2^{10n} \neq O(2^n)$

Proof : by contradiction. If $2^{10n} = O(2^n)$ then there exist constants $c, n_0 > 0$ such that

$$2^{10n} \leq c \cdot 2^n \quad n \geq n_0$$

But then [cancelling $2^n$ ]

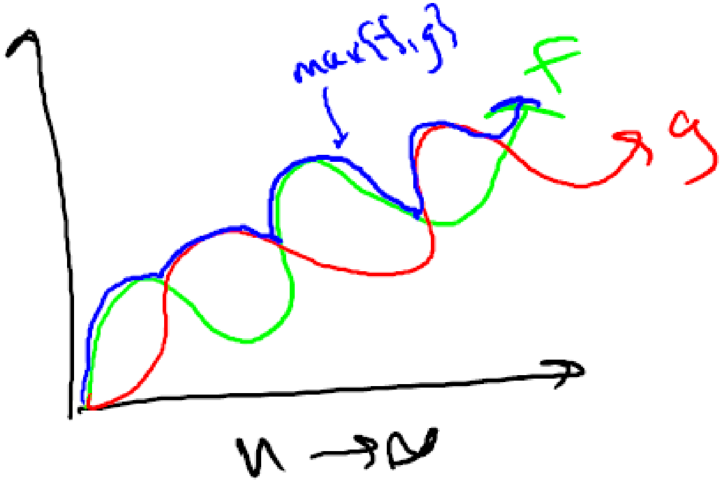
Which is certainly false.

Q.E.D

# Example #3

<u>Claim</u> : for every pair of (positive) functions f(n), g(n),

$$max\{f, g\} = \theta(f(n) + g(n))$$



# Example #3 (continued)

Proof : $max\{f, g\} = \theta(f(n) + g(n))$

For every n, we have

$$max\{f(n), g(n)\} \leq f(n) + g(n)$$

And

$$2 * max\{f(n), g(n)\} \geq f(n) + g(n)$$

Thus $$\frac{1}{2} * (f(n) + g(n)) \leq max\{f(n), g(n)\} \leq f(n) + g(n) \quad \forall n \geq 1$$

$$=> max\{f, g\} = \theta(f(n) + g(n)) \quad [where\ n_0 = 1, c_1 = 1/2, c_2 = 1]$$

Q.E.D