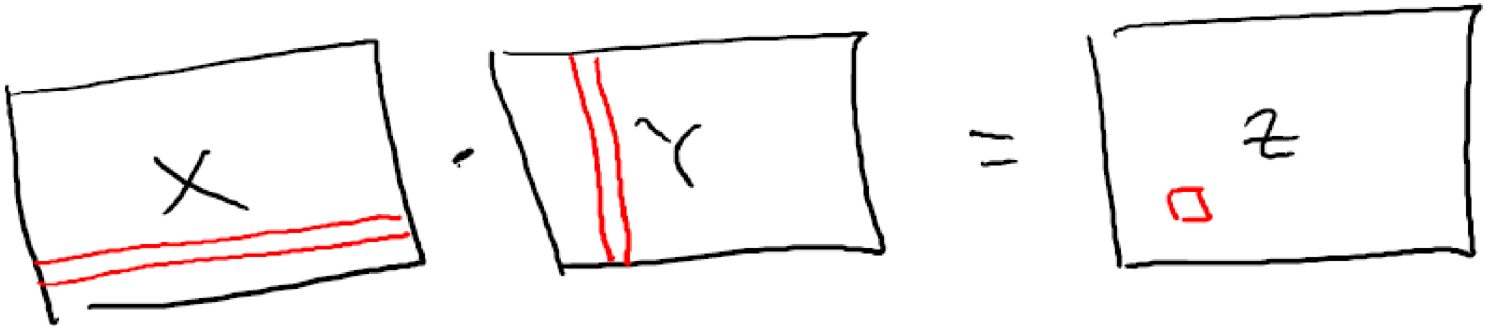
Divide and Conquer

Matrix mulitplication

Design and Analysis of Algorithms I

Matrix Multiplication



(all n X n matrices)

Where z_{ij} = (ith row of X) . (jth column of Y)

$$= \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$$

Note : input size $= \theta(n^2)$

Nextcore AI - Gopal Shangari

EXAMPLE (N=2)

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$$z_{ij} = \sum_{k=1}^{n} X_{ik} \cdot Y_{kj}$$

$\theta(n)$

What is the asymptotic running time of the straightforward iterative algorithm for matrix multiplication?

- $\theta(n \log n)$
- $\theta(n^2)$
- $\theta(n^3)$
- $\theta(n^4)$

THE DIVIDE AND CONQUER PARADIGM

1. DIVIDE into smaller subproblems

2. CONQUER subproblems recursively.

3. COMBINE solutions of subproblems into one for the original problem.

APPLYING DIVIDE AND CONQUER

Idea :

Write $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

[where A through H are all n/2 by n/2 matrices]

Then : (you check)

$$X \cdot Y = \begin{pmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{pmatrix}$$

RECURSIVE ALGORITHM #1

Step 1 : recursively compute the 8 necessary products.

Step 2 : do the necessary additions $(\theta(n^2)\ time)$

Fact : runtime is $\theta(n^3)$ [follows from the master method]

Strassen's Algorithm (1969)

Step 1 : recursively compute only 7 (cleverly chosen) products

Step 2 : do the necessary (clever) additions + subtractions (still $\theta(n^2)$ time)

Fact : better than cubic time!

[see Master Method lecture]

$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ $\quad Y = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$

The Details

The Seven Products : $P_1 = A(F-H)$, $P_2 = (A+B)H$, $P_3 = (C+D)E$, $P_4 = D(G-E)$, $P_5 = (A+D)(E+H)$, $P_6 = (B-D)(G+H)$, $P_7 = (A-C)(E+F)$

Claim : $X \cdot Y = \begin{pmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{pmatrix} = \begin{pmatrix} P_5+P_4-P_2+P_6 & P_1+P_2 \\ P_3+P_4 & P_1+P_5-P_3-P_7 \end{pmatrix}$

Proof: $AE + AH + DE + DH + DG - DE - AH - BH + BG + BH - DG - DH = AE + BG$

Q.E.D

Question : where did this come from ? (remains open!)